Bull. Cal. Math. Soc., **114,** (2) 115–124 (2022)

REES MATRIX SEMIGROUP OVER A Γ-SEMIGROUP

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(*Received 28 July 2021 and revision received 23 February 2022*)

Abstract. In this paper, we introduce Rees matrix Γ-semigroup which is a generalization of Ree's matrix semigroups and discuss several of it's properties. Further we describe the amalgam of Rees matrix Γ-semigroups and its embeddability.

2010 Mathematics Subject classification : Primary 06F99; Secondary 06F05

Keywords and Phrases : Γ-semigroups,Semigroup Amalgam,simple,Rees matrix semigroups.

1. Preliminaries. A semigroup is a non-empty set together with an associative binary operation. Let S be a semigroup, I and Λ non-empty sets and P be a $I \times \Lambda$ matrix with entries $p_{i,\lambda}$ taken from S. Then $M(S; I, \Lambda; P)$ is the set $I \times S \times \Lambda$ together with the multiplication $(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda,j}t, \mu)$ is a semigroup called Rees matrix semigroup over a semigroup S. Γ-semigroups are one class of generalized semigroups introduced by M K Sen (Sen and Saha, 1986). Let M and Γ are two non-empty sets. Then M is a Γ-semigroup if the following axioms are satisfied : for all $m_1, m_2, m_3 \in M$ and $\alpha, \beta \in \Gamma$

- $m_1 \alpha m_2 \in M$.
- $(m_1 \alpha m_2)\beta m_3 = m_1 \alpha (m_2 \beta m_3)$.

A subset K of the set M is called a Γ-subsemigroup if $K\Gamma K \subset K$. An element m in a Γ-semigroup M is called α-idempotent if $m\alpha m = m$ for some $\alpha \in \Gamma$, if every element in the Γ-semigroup M is an α-idempotent for some $\alpha \in \Gamma$ then M is called an idempotent Γ-semigroup. For a Γ-semigroup M, a subset $A \subset M$ is called left(right) Γ-ideal if $M\Gamma A \subset A$ ($A\Gamma M \subset A$). The smallest left Γ-ideal containing $m \in M$ is called principal left ideal generated by m and is denoted by $(m)_l$. Similarly the smallest right Γ-ideal containing $m \in M$ is called principal right ideal generated by m and is denoted by $(m)_r$. The Green's relations in Γ-semigroups are as follows :

- $m \mathscr{L} n$ if and only if $(m)_l = (n)_l$

- $m\mathscr{R}n$ if and only if $(m)_r = (n)_r$
- $m\mathscr{J}n$ if and only if $(m) = (n)$
- $m\mathscr{H}n$ if and only if $m\mathscr{L}n$ and $m\mathscr{R}n$
- $m\mathscr{D}n$ if and only if there exist p such that $m\mathscr{L}p$ and $p\mathscr{R}n$

A Γ-semigroup M is said to be left(right) simple if it has no proper left(right) ideals and is said to be simple if it has no proper ideals. Let M be a Γ-semigroup, an element $m \in M$ is said to be Γ-regular if $m = m\Gamma M\Gamma m$ and if every element m of M is Γ-regular then M is called Γ-regular semigroup.

2. Rees Matrix Semigroup over a Γ-semigroup. Let G be a group, I, Λ be indexing sets and P be a $\Lambda \times I$ sandwich matrix over $G \cup \{0\}$ then $\mathscr{M} = \mathscr{M}^0(G; I; \Lambda; P) = \{(i, a, \lambda) \in I \times G \times \Lambda\} \cup \{0\}$ with binary composition $(i, a, \lambda).(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$ is called the Rees matrix semigroup. Analogous to this we proceed to discuss a Rees matrix semigroup over a Γ-semigroup M.

DEFINITION 2.1 *Let M be a Γ-semigroup, let I, Λ be non-empty sets and $P = [p_{\lambda,i}]$ be a $\Lambda \times I$ sandwich matrix with entries from Γ. Then $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$, the set of all $I \times \Lambda$ matrices over M, is a Γ-semigroup with operation defined by*

$$A_1 \cdot A_2 = A_1 P A_2$$

where the operation on the right is the usual matrix multiplication. Then $(\mathscr{R}_\Gamma, \cdot)$ is called Rees matrix semigroup over a Γ-semigroup M and will call a Rees matrix Γ-semigroup.

As in the case of semigroup, the sandwich matrix P is said to regular if each column and each row contains atleast one nonzero entry from Γ.

EXAMPLE 2.2 *Let $M = \{0, i, -i\}$ and $\Gamma = \{0, i, -i\}$. Then M is a Γ-semigroup and $\mathscr{R}_\Gamma = \mathcal{M}(M, I; \Lambda, P)$, be the set of all $I \times \Lambda$ matrices over M and P be a $\Lambda \times I$ sandwich matrix over the set Γ with entries from Γ. Then for all $M_1, M_2 \in \mathscr{R}_\Gamma$ with $M_1 \cdot M_2 = M_1 P M_2$, where the multiplication on the right is the usual multiplication of matrices. $(\mathscr{R}_\Gamma, \cdot)$ is a Rees matrix Γ-semigroup.*

EXAMPLE 2.3 *Let S be the set of all non-positive integers and Γ the set of all non-positive even numbers, such that S is a Γ-semigroup with usual multiplication of numbers. Consider P as an $\Lambda \times I$ matrix over Γ. Then the Rees matrix Γ-semigroup*

$\mathscr{R}_\Gamma = \mathcal{S}(S, I; \Lambda, P)$ *consists of all matrices of the form* $I \times S \times \Lambda$ *with multiplication* $s_1 \cdot s_2 = s_1 P s_2$ *for each* $s_1, s_2 \in \mathscr{R}_\Gamma$.

In the following we discuss the regular elements and idempotents of a Rees matrix Γ-semigroup.

DEFINITION 2.4 *Let* M *be a* Γ*-semigroup. An element* m *of* M *is called* α*-idempotent if* $m\alpha m = m$ *for some* $\alpha \in \Gamma$. *If every element in the* Γ*-semigroup* M *is an* α*-idempotent for some* $\alpha \in \Gamma$ *then* M *is called an idempotent* Γ*-semigroup.*

DEFINITION 2.5 *Let* $\mathscr{R}_\Gamma = \mathscr{M}(M, I; \Lambda, P)$ *be a Rees matrix* Γ*-semigroup over a* Γ*-semigroup* M *with indexing sets* I, Λ. *An element* $(i, m, \lambda) \in \mathscr{R}_\Gamma$, *where* $i \in I, \lambda \in \Lambda$ *and* $m \in M$, *is called an idempotent if* $(i, m, \lambda) \cdot (i, m, \lambda) = (i, m, \lambda)$.

THEOREM 2.6 *Let* M *be a* Γ*-semigroup and* $\mathscr{R}_\Gamma = \mathscr{M}(M, I; \Lambda, P)$ *be Rees matrix* Γ*-semigroup. An element* $(i, m, \lambda) \in \mathscr{R}_\Gamma$ *is an idempotent if and only if* m *is a* $p_{\lambda,i}$*-idempotent in* M.

Proof: Assume an element $(i, m, \lambda) \in \mathscr{R}_\Gamma$ is an idempotent . Then

$$(i, m, \lambda) \cdot (i, m, \lambda) = (i, m, \lambda)$$
$$(i, m p_{\lambda,i} m, \lambda) = (i, m, \lambda)$$

that is,

$$m p_{\lambda,i} m = m.$$

Hence m is a $p_{\lambda,i}$-idempotent in M.

Conversely suppose that m is a $p_{\lambda,i}$-idempotent in M. Then

$$m p_{\lambda,i} m = m$$
$$(i, m p_{\lambda,i} m, \lambda) = (i, m, \lambda)$$
$$(i, m, \lambda) \cdot (i, m, \lambda) = (i, m, \lambda)$$

Therefore, (i, m, λ) is an idempotent in $\mathscr{R}_\Gamma$.

EXAMPLE 2.7 *Let* $K = \{0, i, -i\}$, *consider* Γ *as* K *itself such that* K *is a* Γ*-semigroup. Let* $I = \{1\}$, $\Lambda = \{1, 2\}$ *be indexing sets and* $P = \begin{bmatrix} 0 \\ -i \end{bmatrix}$ *be a* 2×1 *matrix over* Γ. *Then* $\mathscr{R}_\Gamma = \{(1, i, 1), (1, i, 2), (1, 0, 1), (1, 0, 2), (1, -i, 1), (1, -i, 2)\}$. *Since* $i(-i)i = i$ *and* $(-i)i(-i) = (-i)$, i *is a* $-i$*-idempotent and* $-i$ *is an* i*-idempotent of* K *and* $(1, i, 2)(1, i, 2) = (1, i p_{2,1} i, 2) = (1, i(-i)i, 2) = (1, i, 2)$, *hence* $(1, i, 2)$ *is an idempotent in* $\mathscr{R}_\Gamma$.

We characterize regular elements of Rees matrix Γ-semigroup as follows:

DEFINITION 2.8 *Let M be a Γ-semigroup. An element $m \in M$ is called (α, β)-regular in M for $\alpha, \beta \in \Gamma$ if there is an element $x \in M$ such that $m = m\alpha x\beta m$.*

THEOREM 2.9 *Let $\mathscr{R}_\Gamma = \mathscr{M}(K, I; \Lambda, P)$ be a Rees matrix Γ-semigroup over K where I, Λ indexing sets and P be a $\Lambda \times I$ matrix over Γ. An element $(i, x, \lambda) \in \mathscr{R}_\Gamma$ is regular if and only if x is a $(p_{\lambda,j}, p_{\mu,i})$-regular element of K.*

Consider an element $(i, x, \lambda) \in \mathscr{R}_\Gamma$. Then (i, x, λ) is a regular element of $\mathscr{R}_\Gamma$ if there exist an element (j, y, μ) such that

$$\begin{aligned} (i, x, \lambda)(j, y, \mu)(i, x, \lambda) &= (i, x, \lambda) \\ (i, xp_{\lambda,j}y, \mu)(i, x, \lambda) &= (i, x, \lambda) \\ (i, xp_{\lambda,j}yp_{\mu,i}x, \lambda) &= (i, x, \lambda) \\ xp_{\lambda,j}yp_{\mu,i}x &= x \end{aligned}$$

hence x is a $(p_{\lambda,j}, p_{\mu,i})$-regular element of K.

Conversely suppose that x is a $(p_{\lambda,j}, p_{\mu,i})$-regular element of K. Then $xp_{\lambda,j}yp_{\mu,i}x = x$ and $(i, xp_{\lambda,j}yp_{\mu,i}x, \lambda) = (i, x, \lambda)$.

Also $(i, x, \lambda)(j, y, \mu)(i, x, \lambda) = (i, x, \lambda)$. Hence (i, x, λ) is a regular element of $\mathscr{R}_\Gamma$.

EXAMPLE 2.10 *Consider the Rees matrix Γ-semigroup $\mathscr{R}_\Gamma = \mathscr{M}(K, I; \Lambda, P)$ where $K = \{0, i, -i\}$, Γ is taken as K itself such that K is a Γ-semigroup, $I = \{1\}$, $\Lambda = \{1, 2\}$ be indexing sets with sandwich matrix $P = \begin{bmatrix} i \\ -i \end{bmatrix}$. In K, $i(i)(-i)(-i)i = i$ and hence i is an $(i, -i)$-regular element of K. Now for $(1, i, 1)$, there exist $(1, -i, 2)$ such that $(1, i, 1)(1, -i, 2)(1, i, 1) = (1, i, 1)$ and hence $(1, i, 1)$ is an $(i, -i)$-regular element in $\mathscr{R}_\Gamma$.*

3. Hereditary properties on Rees matrix Γ-semigroup. In this section we discuss some results on properties of Rees matrix Γ-semigroup $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$ which inherited from the Γ-semigroup M.

THEOREM 3.1 *A Rees matrix Γ-semigroup $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$ over a Γ semigroup M with $|I| = 1$ is right simple if and only if M is right simple Γ semigroup.*

Proof: Let $\mathscr{R}_\Gamma$ be an arbitrary Rees matrix Γ-semigroup over a Γ semigroup M and assume the Γ semigroup M is right simple.

Let $(i, m, \lambda), (i, n, \mu)$ be arbitrary elements in $\mathscr{R}_\Gamma$. Then,

$$m\Gamma M = M \Rightarrow mp_{\lambda,i}M = M$$

implies $mp_{\lambda,i}m' = n$ for some $m' \in M$.

Hence,

$$(i, m, \lambda)(i, m', \mu) = (i, mp_{\lambda,i}n', \mu) = (i, n, \mu).$$

Then, $(i, m, \lambda)\mathscr{M} = \mathscr{M}$ for all $(i, m, \lambda) \in \mathscr{M}$. Thus $\mathscr{M}$ is right simple.

Conversely, suppose that $\mathscr{M}$ is right simple.

Let $m, n \in M$ be arbitrary elements and $p_{\lambda,i} = \alpha \in \Gamma$.

Then for any $\lambda \in \Lambda$, we have $(i, m, \lambda)\mathscr{M} = \mathscr{M}$ and so,

$$(i, m, \lambda)(i, m', \eta) = (i, n, \eta)$$

for some $(i, m', \eta) \in \mathscr{M}$. Hence

$$mp_{\lambda,i}m' = n.$$

Thus $m\Gamma M = M$ for all $m \in M$ and the Γ-semigroup M is right simple.

THEOREM 3.2 *A Rees matrix* Γ*-semigroup* $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$ *over a* Γ *semigroup* M *with* $|I| = 1$ *is simple if and only of* M *is a simple* Γ *semigroup.*

Proof: Let $\mathscr{R}_\Gamma$ be an arbitrary Rees matrix Γ-semigroup over a Γ semigroup M.

Assume that the Γ semigroup M is simple. Let $(i, a, \lambda), (i, b, \mu) \in \mathscr{M}(M; I; \Lambda; P)$ be arbitrary elements. Now for arbitrary $\rho \in \Gamma$,

$$Mp_{\rho,i}ap_{\lambda,i}M = M,$$

and so

$$a'p_{\rho,i}ap_{\lambda,i}b' = b$$

for some $a', b' \in M$. Hence

$$(i, a', \rho)(i, a, \lambda)(i, b', \mu) = (i, a'p_{\rho,i}ap_{\lambda,i}b', \mu) = (i, b, \mu).$$

Thus

$$\mathscr{M}(i, a, \lambda)\mathscr{M} = \mathscr{M}$$

for every $(i, a, \lambda) \in \mathscr{M}$. Then $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$is simple.

Conversely, assume that $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$is simple. Let $m, n \in M$ and $\alpha, \beta \in \Lambda$ be arbitrary elements. Then

$$\mathscr{M}(i, m, \alpha)\mathscr{M} = \mathscr{M},$$

and so there are elements $(i, m', \rho), (i, n', \mu) \in \mathscr{M}(M; I; \Lambda; P)$ such that

$$(i, m' p_{\rho,i} m p_{\lambda,i} n', \mu) = (i, m', \rho)(i, m, \lambda)(i, n', \mu) = (i, n, \mu).$$

Hence

$$m' p_{\rho,i} m p_{\lambda,i} n' = n.$$

Thus $M\Gamma m\Gamma M = M$ for every $m \in M$. Then the Γ semigroup M is simple.

THEOREM 3.3 *A Rees matrix* Γ*-semigroup* $\mathscr{R}_\Gamma = \mathscr{M}(M, I; \Lambda, P)$ *over a* Γ *semigroup* M *with* $|\Lambda| = 1$ *is left simple if and only of* M *is left simple* Γ *semigroup.*

Proof: The proof follows from the same arguments that we discussed in the proof of theorem 3.1.

THEOREM 3.4 *Let* $\mathscr{R}_\Gamma = \mathscr{M}(M, I; \Lambda, P)$ *be Rees matrix* Γ*-semigroup with* $|I| = 1$ *over a* Γ*-semigroup* M*. Then* $\mathscr{R}_\Gamma$ *is left cancellative with* $|\Gamma| = 1$ *if and only if* M *is a left cancellative* Γ*-semigroup.*

Proof: First assume that the Rees matrix Γ-semigroup $\mathscr{M}(M, I; \Lambda, P)$ over a Γ-semigroup M is left cancellative. Let $m_1, m_2, m_3 \in M$ and $\alpha \in \Gamma$ be arbitrary element with $m_1 \alpha m_2 = m_1 \alpha m_3$. Then for arbitrary $\lambda \in \Lambda$,

$$(i, m_1, \gamma)(i, m_2, \lambda) = (i, m_1, \gamma)(i, m_3, \lambda).$$

Since the Rees matrix Γ semigroup $\mathscr{M}$ is cancellative , we get

$$(i, m_2, \lambda) = (i, m_3, \lambda)$$

and so $m_2 = m_3$ and hence M is a cancellative Γ-semigroup.

Conversely, assume that, for every $m_1, m_2, m_3 \in M$ and $\alpha \in \Gamma$, the assumption $m_1 \alpha m_2 = m_1 \alpha m_3$ implies $m_2 = m_3$. Let $(i, m_1, \gamma), (i, m_2, \lambda), (i, m_3, \tau) \in \mathscr{M}(M; I; \Lambda; P)$ be arbitrary elements with

$$(i, m_1, \gamma)(i, m_2, \lambda) = (i, m_1, \gamma)(i, m_3, \tau)$$

then

$$(i, m_1 p_{\gamma,i} m_2, \lambda) = (i, m_1 p_{\gamma,i} m_3, \tau),$$

that is, $m_1 p_{\gamma,i} m_2 = m_1 p_{\gamma,i} m_3$ and $\lambda = \tau$. Since $|\Gamma| = 1$ and M is left cancellative, we get $m_2 = m_3$, and so $(i, m_2, \lambda) = (i, m_3, \tau)$. Hence the Rees matrix Γ-semigroup $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$ is left cancellative.

Analogously we have the following theorem:

THEOREM 3.5 *Let* $\mathscr{R}_\Gamma = \mathscr{M}(M; I; \Lambda; P)$ *be Rees matrix* Γ*-semigroup with* $|\Lambda| = 1$ *over a* Γ*-semigroup* M*. The Rees matrix* Γ*-semigroup* $\mathscr{R}_\Gamma$ *is right cancellative with* $|\Gamma| = 1$ *if and only if* M *is a right cancellative* Γ*-semigroup.*

4. Embedding on Rees matrix Γ-semigroup and Amalgam. In this section we discuss conditions for embedding of a Rees matrix Γ-semigroup to another Rees matrix Γ-semigroup. Also the amalgam of Rees matrix Γ-semigroup is defined and discussing its embeddability.

THEOREM 4.1 *Let* S *be a* Γ*-semigroup and* T *a* $\Gamma^{'}$*-semigroup.* $\Phi = (\psi, \phi)$ *an embedding of* (S, Γ) *into* $(T, \Gamma^{'})$*. For non-empty sets* I *and* Λ*, the Rees matrix* Γ*-semigroup* $\mathscr{M}(S; I; \Lambda; P)$ *is embeddable into the Rees matrix* $\Gamma^{'}$*-semigroup* $\mathscr{M}(T; I; \Lambda; P')$ *where* $P' = (\phi(p_{\lambda,i}))$ *for every* $i \in I$ *and* $\lambda \in \Lambda$*.*

Proof: We claim the mapping $\Upsilon : (i, s\gamma s', \lambda) \mapsto (i, \Phi(s\gamma s'), \lambda)$ is an embedding of the Rees matrix Γ-semigroup $\mathscr{M}(S; I; \Lambda; P)$ into the Rees matrix $\Gamma^{'}$-semigroup $\mathscr{M}(T; I; \Lambda; P')$ where $P' = (\phi(p_{i\lambda}))$ for every $s \in S$, $i \in I$,$\gamma \in \Gamma$ and $\lambda \in \Lambda$.

First, the mapping Υ is injective, for

$$\Upsilon((i, s, \lambda)) = \Upsilon((j, s', \mu))$$

where $s, s' \in S$, $i, j \in I$ and $\lambda, \mu \in \Lambda$. Then

$$(i, \psi(s), \lambda) = (j, \psi(s'), \mu) \Longrightarrow \psi(s) = \psi(s')$$

and $i = j$, $\lambda = \mu$. Since ψ is injective, we get

$$(i, s, \lambda) = (j, s', \mu).$$

Further the mapping Υ is Γ-homomorphism for $(i, s, \lambda), (j, s', \mu) \in \mathscr{M}(S; I; \Lambda; P)$ be arbitrary elements. Then

$$\begin{aligned} \Upsilon((i, s, \lambda)(j, s', \mu)) &= \Upsilon((i, sp_{\lambda,j}s', \mu)) \\ &= (i, \Phi(sp_{\lambda,j}s'), \mu) \\ &= (i, \psi(s)\phi(p_{\lambda,j})\psi(s'), \mu) \\ &= (i, \psi(s), \lambda)(j, \psi(s'), \mu) \\ &= \Upsilon((i, s, \lambda))\Upsilon((j, s', \mu)) \end{aligned}$$

that is, Υ is a Γ-homomorphism. Consequently the mapping Υ is an embedding.

In a more general way we have the following

THEOREM 4.2 *Let S be a Γ-semigroup and T be a Γ'-semigroup. S_1 be Rees matrix Γ-semigroup $\mathscr{M}(S, I; \Lambda, P)$ with sandwich matrix P whose entries are from Γ and S_2 be Rees matrix Γ-semigroup $\mathscr{M}(T, J; \Pi, Q)$ with sandwich matrix Q whose entries are from Γ'. Then S_1 and S_2 are isomorphic if and only if there exist a (Γ, Γ')-isomorphism $\Delta = (\delta_1, \delta_2)$ where $\delta_1 : S \to T$, $\delta_2 : \Gamma \to \Gamma'$, bijections $\epsilon : I \to J$, $\zeta : \Lambda \to \Pi$ and $Q = (\delta_2(p_{\lambda,i}))$ for every $i \in I$ and $\lambda \in \Lambda$.*

DEFINITION 4.3 *An amalgam of Rees matrix Γ-semigroups consists of non-empty sets S_1, S_2, U and Γ such that S_1, S_2, U are Γ-semigroups over the same Γ and $\phi_i : U \to S_i$ for $i = \{1, 2\}$ be Γ-monomorphisms. For non-empty indexing sets I, J and $J \times I$ sandwich matrices P, Q, R over the set Γ the Rees matrix Γ-semigroup $M = \mathscr{M}(U, I; J, P)$, $M_1 = \mathscr{M}(S_1, I; J, Q)$, $M_2 = \mathscr{M}(S_2, I; J, R)$ constitutes a Rees matrix Γ-semigroup amalgam $[M, M_1, M_2, \psi_1, \psi_2]$ with Γ-monomorphisms $\psi_i : M \to M_i$ for $i = \{1, 2\}$.*

The Rees matrix Γ-semigroup amalgam $[M, M_1, M_2, \psi_1, \psi_2]$ is said to be embeddable in a Rees matrix Γ-semigroup $M_3 = \mathscr{M}(T, I; J, W)$ with sandwich matrix W over Γ if there exist mappings $\eta : M \to M_3$ and $\eta_i : M_i \to M_3$ such that (i) $\psi_1\eta_1 = \psi_2\eta_2 = \eta$. Then the amalgam is said to be weakly embeddable. In addition, if the amalgam satisfies $\eta_1(M_1) \cup \eta_2(M_2) = \eta(M)$ then the amalgam is called strongly embeddable.

THEOREM 4.4 *Let $\mathscr{U} = [U, S_1, S_2, \phi_1, \phi_2]$ be Γ-semigroup amalgam where U, S_1, S_2 are Γ-semigroups over the same Γ, ϕ_1, ϕ_2 be Γ-monomorphisms from U to S_1, S_2 respectively and $\mathscr{U}$ be embeddable in a Γ-semigroup T. Then the Rees matrix Γ-semigroup amalgam $[M, M_1, M_2, \psi_1, \psi_2]$ is embeddable in $\mathscr{M}(T, I; J, P)$*

Proof: Consider the Γ-semigroup amalgam $\mathscr{U} = [U, S_1, S_2, \phi_1, \phi_2]$ and let $\phi_1(u) = s_1$, $\phi_2(u) = s_2$ where $u \in U$, $s_1 \in S_1$ and $s_2 \in S_2$. Since $\mathscr{U}$ is embeddable in a Γ-semigroup T, there exist mappings $\theta_1 : S_1 \to T$ and $\theta_2 : S_2 \to T$ such that $\phi_1\theta_1 = \phi_2\theta_2$ and $\theta_1(s_1) = \theta_2(s_2)$ hence there exists $u \in U$ such that $s_1 = \phi_1(u)$ and $s_2 = \phi_{2(u)}$ where $s_1 \in S_1$ and $s_2 \in S_2$.
Consider the Rees matrix Γ-semigroups M, M_1 and M_2 with $\psi_1 : M \to M_1$ and $\psi_2 : M \to M_2$, let $(i, u, j) \in M = \mathscr{M}(U, I; J, P)$. Define $\psi_1 : M \to M_1$ by

$$\psi_1(i, u, j) = (i, \phi_1(u), j).$$

Since ϕ_1 is a Γ-monomorphism, ψ_1 is also a Γ-monomorphism.

Next we construct a Rees matrix Γ-semigroup over T where the amalgam $\mathscr{U} = [U, S_1, S_2, \phi_1, \phi_2]$ is embeddable. Let it be $M_T = \mathscr{M}(T, I; J, P)$ where the sandwich matrix P is over Γ. Since the amalgam $\mathscr{U}$ is embeddable in T, M_T has the property that $(i, \phi_1\theta_1(u), j) = (i, \phi_2\theta_2, j)$ and $(i_1, \theta_1(s_1), j_1) = (i_2, \theta_2(s_2), j_2)$ implies $i_1 = i_2$, $j_1 = j_2$ and $\exists u \in U$ such that $s_1 = \phi_1(u)$ and $s_2 = \phi_{2(u)}$ where $s_1 \in S_1$ and $s_2 \in S_2$.

Now let $(i, u, j) \in M_T = \mathscr{M}(T, I; J, Z)$. Define the mapping $\eta_1 : M_1 \to M_T$ by $\eta_1(i, s_1, j) = (i, \theta_1(s_1), j)$ and $\eta_2 : M_2 \to M_T$ by $\eta_2(i, s_2, j) = (i, \theta_2(s_2), j)$. To show that the mappings η_1 and η_2 are Γ-monomorphisms, consider two arbitrary elements (i, s_1, j) and $(i, s_1^{'}, j) \in M_1$ and

$$\begin{aligned} \eta_1((i, s_1, j)) &= \eta_2((i, s_1^{'}, j)) \\ (i, \theta_1(s_1), j) &= (i, \theta_2(s_1^{'}), j) \\ \theta_1(s_1) &= \theta_2(s_1^{'}) \end{aligned}$$

since θ_1 is one-one, we get $s_1 = s_1'$. Thus $(i, s_1, j) = (i, s_1^{'}, j)$. Also

$$\begin{aligned} \eta_1((i, s_1, j)(i, s_1^{'}, j)) &= \eta_1((i, s_1 q_{ji} s_1^{'}, j)) \\ &= (i, \theta_1(s_1 q_{ji} s_1^{'}), j) \\ &= (i, \theta_1(s_1, j)(i, s_1^{'}), j) \\ &= \eta_1((i, s_1, j)\eta_1(i, s_1^{'}, j). \end{aligned}$$

Thus the mapping η_1 is a Γ-monomorphism and similarly η_2. Hence η_1, η_2 are embeddings. Now to verify the conditions of embeddability, consider arbitrary element (i, u, j) and

$$\begin{aligned} \eta_1\psi_1((i, u, j)) &= \eta_1(\psi_1(i, u, j)) = \eta_1((i, \phi_1(u), j)) \\ &= \eta_1((i, s_1, j)) = (i, \theta_1(s_1), j) \\ &= (i, \theta_1\phi_1(u), j) = (i, \theta_2\phi_2(u), j) \\ &= (i, \theta_2(s_2), j) = \eta_2((i, s_2, j)) \\ &= \eta_2((i, \phi_2(u), j)) = \eta_2(\psi_2(i, u, j)) \\ &= \eta_2\psi_2((i, u, j)) \end{aligned}$$

hence $\eta_1\psi_1 = \eta_2\psi_2$.

Also assume $\eta_1((i, s_1, j)) = \eta_2((i, s_2, j))$, then $(i, \theta_1(s_1), j) = (i, \theta_2(s_2), j)$ implies $\theta_1(s_1) = \theta_2(s_2)$. Since $\mathscr{U}$ is embeddable there exist $u \in U$ such that $\phi_1(u) = s_1$

and $\phi_2(u) = s_2$. Therefore, we get, there exist an element (i, u, j) such that $\psi_1((i, u, j)) = (i, s_1, j)$ and $\psi_2((i, u, j)) = (i, s_2, j)$. Thus the amalgam satisfies both the conditions and as a result it is embeddable in M_T.

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